Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 2

31 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) Let N be North

$N\hat{J}D = 34°$ OR $D\hat{J}L = 56°$ (must be labelled or indicated in diagram): ***(A1)***

$J\hat{D}L = 99(°)$ ***A1***

> **Note:** Accept $\frac{11\pi}{20}, 1.73$ (radians).

[2 marks]

(b) attempt to apply the sine rule ***(M1)***

$\frac{DL}{\sin 56°} = \frac{500}{\sin 99°}$ OR $\frac{DL}{\sin 0.977384...} = \frac{500}{\sin 1.72787...}$ ***(A1)***

$419.685\ldots$

$DL = 420$ (km) ***A1***

> **Note:** Award ***M1A1A0*** for 261 (km) from use of degrees with GDC set in radians (with or without working).

[3 marks]

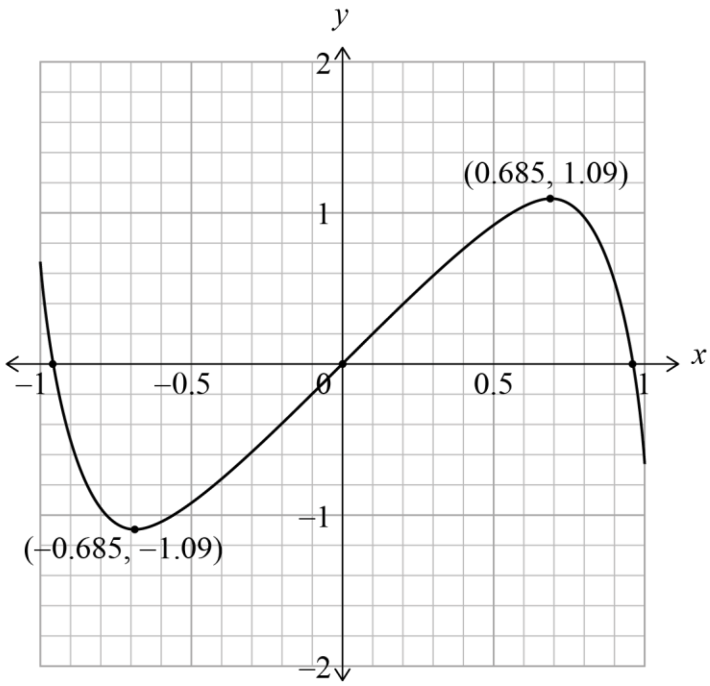
Total [5 marks]

2. (a) attempt to substitute g into f ***(M1)***

$(f \circ g)(x) = 2\tan x - \tan^3 x$ ***A1***

[2 marks]

(b)



A1A1A1

Note: ***A1*** for approximately correct odd function passing through the origin with a maximum above $y=1$ and a minimum below $y=-1$.

A1 for endpoints at $x=\pm 1$ and y in the intervals $[0.6, 0.8]$ and $[-0.8, -0.6]$

A1 for maximum in approximately correct position and labelled $(0.685, 1.09)$ AND minimum in approximately correct position and labelled $(-0.685, -1.09)$. For approximate position, allow $-0.8 \le x \le -0.6$, $-1.2 \le y \le -1$ for minimum and $0.6 \le x \le 0.8$, $1 \le y \le 1.2$ for maximum. If the candidate gives the coordinates of extrema below their sketch, only award this mark if extrema are marked in the correct interval (eg by a dot).

[3 marks]

Total [5 marks]

3. (a) recognising to find $y(25)$ ***(M1)***

$y(25) = -0.6 \times 25^2 + 23 \times 25 + 110$

$= 310$ (children) ***A1***

[2 marks]

(b) recognizing x on y is required ***(M1)***

$0.0935114\ldots$ and $7.43053\ldots$ ***(A1)***

$x = 0.0935y + 7.43$ ***A1***

[3 marks]

(c) attempt to substitute their answer to part (a) into their regression equation for either x or y ***(M1)***

$x = 0.0935114... \times 310 + 7.43053... (= 36.4190...)$

36 (accept 37 or 36.4) ***A1***

Note:	Award ***(M1)A1FT*** for $x = 37$ found from using $y = 9.39x - 41.5$.
	Award ***(M1)A0FT*** for a correct ***FT*** answer that lies outside $[15, 46]$.

[2 marks]

Total [7 marks]

4. **METHOD 1**

$Q_1 = 31.86$ OR $Q_3 = 32.14$ ***(A1)***

recognition that the area under the normal curve below Q_1 or above Q_3 is 0.25 OR the area between Q_1 and Q_3 is 0.5 (seen anywhere including on a diagram) ***(M1)***

EITHER

equating an appropriate correct normal CDF function to its correct probability (0.25 or 0.5 or 0.75) ***(A2)***

OR

$z = -0.674489...$ OR $z = 0.674489...$ (seen anywhere) ***(A1)***

$-0.674489... = \frac{31.86 - 32}{\sigma}$ OR $0.674489... = \frac{32.14 - 32}{\sigma}$ ***(A1)***

THEN

$0.207564...$

$\sigma = 0.208$ (mm) ***A1***

METHOD 2

recognition that the area under the normal curve below Q_1 or above Q_3 is 0.25 OR the area between Q_1 and Q_3 is 0.5 (seen anywhere including on a diagram) ***(M1)***

$z = -0.674489...$ OR $z = 0.674489...$ ***(A1)***

$(Q_1 =)32 - 0.674489...\sigma$ OR $(Q_3 =)32 + 0.674489...\sigma$ ***(A1)***

$(Q_3 - Q_1 =)2 \times 0.674489...\sigma$

$2 \times 0.674489...\sigma = 0.28$ ***(A1)***

$0.207564...$

$\sigma = 0.208$ (mm) ***A1***

Total [5 marks]

5. product of a binomial coefficient, a power of ax^3 and a power of b seen ***(M1)***

evidence of correct term chosen

for $n = 8$: $r = 2$ (or $r = 6$) OR for $n = 10$: $r = 2$ (or $r = 8$) ***(A1)***

correct equations (may include powers of x) ***A1A1***

$${}^{8}C_{2}a^{2}b^{6} = 448 \ \left(28a^{2}b^{6} = 448 \Rightarrow a^{2}b^{6} = 16\right), \ {}^{10}C_{2}a^{2}b^{8} = 2880 \ \left(45a^{2}b^{8} = 2880 \Rightarrow a^{2}b^{8} = 64\right)$$

attempt to solve their system in a and b algebraically or graphically ***(M1)***

$b = 2$; $a = \frac{1}{2}$ ***A1A1***

Note: Award a maximum of ***(M1)(A1)A1A1(M1)A1A0*** for $b = \pm 2$ and/or $a = \pm\frac{1}{2}$.

[7 marks]

6. (a) attempt to use De Moivre's theorem ***(M1)***

$\left(\cos\frac{11\pi}{18}+\mathrm{i}\sin\frac{11\pi}{18}\right)^n=\cos\frac{11\pi n}{18}+\mathrm{i}\sin\frac{11\pi n}{18}\left(=\mathrm{e}^{\frac{11\pi n}{18}\mathrm{i}}\right)$ OR $\cos(110^\circ n)+\mathrm{i}\sin(110^\circ n)$

EITHER

attempt to consider imaginary part ***(M1)***

$\sin\frac{11\pi n}{18}=-1$ OR $\sin(110^\circ n)=-1$

OR

attempt to consider argument of $-\mathrm{i}$ ***(M1)***

$\mathrm{e}^{\frac{11\pi n}{18}\mathrm{i}}=\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}$

THEN

$\frac{11\pi n}{18}=\frac{3\pi}{2},\frac{7\pi}{2}\left(,\frac{11\pi}{2}\right)\ldots\left(=\frac{3\pi}{2}+2\pi k,\ k\in\mathbb{Z}\right)$ OR

$110^\circ n=270^\circ,630^\circ\left(,990^\circ\right)\ldots\left(=270^\circ+360^\circ k,\ k\in\mathbb{Z}\right)$ ***(A1)***

$11n=27,63,99,\ldots$

$n=9$ ***A1***

[4 marks]

continued…

Question 6 continued

(b) **EITHER**

$z^{10} = e^{10\left(\frac{11\pi}{18}i\right)} \left(= e^{\frac{55\pi}{9}i} = e^{\frac{\pi}{9}i} \right)$ OR $\arg(z^{10}) = \frac{\pi}{9}$ OR $\arg(z^{10}) = 20^\circ$ ***(A1)***

Note: Accept equivalent arguments given in any interval, in degrees or radians.

recognising that the difference between $\arg(z^{10})$ and $\arg(z)$ is needed ***(M1)***

$\arg(z^{10}) - \arg(z) = \frac{\pi}{9} - \frac{11\pi}{18} = -\frac{\pi}{2}$

OR

recognising that $z^{10} = z^9 \times z$ ***(M1)***

$z^9 = e^{9\left(\frac{11\pi}{18}i\right)} \left(= e^{\frac{11\pi}{2}i} = e^{\frac{3\pi}{2}i} \right)$ OR $\arg(z^9) = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$ OR $\arg(z^9) = 270^\circ$ or -90° ***(A1)***

Note: Accept equivalent arguments given in any interval, in degrees or radians.

THEN

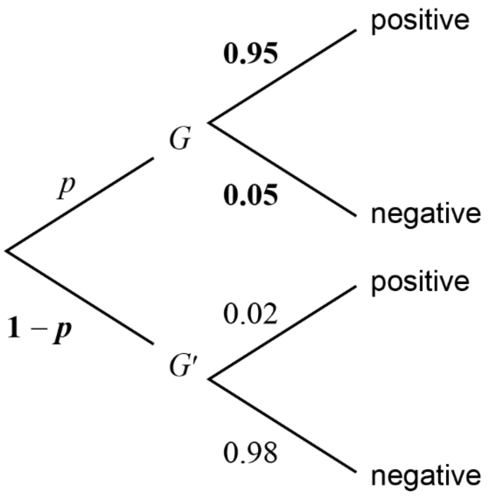
a rotation $\frac{3\pi}{2}$ OR $-\frac{\pi}{2}$ OR equivalent angle about the origin. ***A1***

Note: Accept correct answer given in degrees.
Accept $\frac{\pi}{2}$ clockwise or $\frac{11\pi}{2}$ or $\frac{(4k-1)\pi}{2}$ for $k \in \mathbb{Z}$.
The centre must be stated to gain the final ***A1***.

[3 marks]

Total [7 marks]

7. (a)



A1A1

> **Note:** award ***A1*** for branch correctly labelled $1-p$
> award ***A1*** for branches correctly labelled 0.95 and 0.05
> award ***A0*** for G' branch labelled p'
> award ***A0*** for G' branch labelled q unless explicitly defined as $1-p$

[2 marks]

continued…

Question 7 continued

(b) **METHOD 1**

recognizing conditional probability ***(M1)***

$\mathrm{P}(G'|pos)$ OR $\mathrm{P}(G|pos)$

$\dfrac{0.02(1-p)}{0.95p+0.02(1-p)}\left(=\dfrac{18}{150}\right)$ OR $\dfrac{0.95p}{0.95p+0.02(1-p)}\left(=\dfrac{132}{150}\right)$ ***(A1)(A1)***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$p = 0.133738$

$p = 0.134$ ***A1***

METHOD 2

attempt to set up a system of equations (S = sample size) ***(M1)***

$p(0.95S) = 132$ and $(1-p)(0.02S) = 18$ ***(A1)***

attempt to solve for p or S ***(M1)***

$\dfrac{0.95p}{0.02(1-p)} = \dfrac{132}{18}$

OR $S = pS + (1-p)S = \dfrac{132}{0.95} + \dfrac{18}{0.02} = 138.947... + 900 = 1038.94...$

$p = 0.133738...$

$p = 0.134$ ***A1***

METHOD 3

attempt to find the number of parrots with the gene and the number without ***(M1)***

number of parrots with the gene $\approx \frac{132}{0.95} = 138.947...$ AND

number of parrots without the gene $\approx \frac{18}{0.02} = 900$ ***(A1)***

number of parrots in the sample $\approx 138.947... + 900 = 1038.94...$

attempt to find proportion of sample with the gene ***(M1)***

$p \approx \frac{138.947...}{1038.94...} = 0.133738...$

$p = 0.134$ ***A1***

[4 marks]

Total [6 marks]

8. direction vector of the line is $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ (seen anywhere) ***(A1)***

normal vector of the plane is $\begin{pmatrix} 4 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}$ (seen anywhere) ***(A1)***

EITHER

correct scalar product $12 + 2\cos\alpha - \sin\alpha$ (seen anywhere) ***(A1)***

one correct magnitude (seen anywhere) ***(A1)***

$\sqrt{16 + \cos^2\alpha + \sin^2\alpha}\ \left(= \sqrt{17}\right),\ \sqrt{9+4+1}\ \left(= \sqrt{14}\right)$

recognizing angle between normal and direction vector is $\frac{\pi}{2} - \alpha$ (seen anywhere) ***(M1)***

Note: angle $\frac{\pi}{2} - \alpha$ may be implied by use of $\sin\alpha$ on the RHS of the step below

attempt to substitute into the formula for the angle between two vectors to form an equation in α ***(M1)***

$12 + 2\cos\alpha - \sin\alpha = \sqrt{17}\sqrt{14}\cos\left(\frac{\pi}{2} - \alpha\right)$ OR $12 + 2\cos\alpha - \sin\alpha = \sqrt{17}\sqrt{14}\sin\alpha$

continued…

Question 8 continued

OR

correct expression for the magnitude of the vector product

$\left|\begin{pmatrix} 2\sin\alpha+\cos\alpha \\ -4-3\sin\alpha \\ 3\cos\alpha-8 \end{pmatrix}\right| \left(=\sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2}\right)$ (seen anywhere) ***(A1)***

one correct magnitude (seen anywhere) ***(A1)***

$\sqrt{16+\cos^2\alpha+\sin^2\alpha}\ \left(=\sqrt{17}\right),\ \sqrt{9+4+1}\ \left(=\sqrt{14}\right)$

recognizing angle between normal and direction vector is $\frac{\pi}{2}-\alpha$ (seen anywhere) ***(M1)***

Note: angle $\frac{\pi}{2}-\alpha$ may be implied by use of $\cos\alpha$ on the RHS of the step below

attempt to substitute into the formula for the angle between two vectors to form an equation in α ***(M1)***

$\sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2}=\sqrt{17}\sqrt{14}\sin\left(\frac{\pi}{2}-\alpha\right)$ OR

$\sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2}=\sqrt{17}\sqrt{14}\cos\alpha$

THEN

$\alpha=0.932389...$

$\alpha=0.932$ ***A1***

Note: Award maximum ***(A1)(A1)(A1)(A1)(M1)(M1)A0*** for a correct answer given in degrees $\alpha=54.4219...^\circ$.

[7 marks]

9. Assume $p^2 - 8q - 11 = 0$, $(p, q \in \mathbb{Z})$ ***M1***

Note: This ***M1*** is dependent on the assumption of truth (implied by "assume" or "suppose that … is true".)
Subsequent marks should be awarded independently.

EITHER

$p^2 = 8q + 11(= 2(4q + 5) + 1)$ so p^2 odd $\Rightarrow p$ odd ***R1***

OR

p even $\Rightarrow p^2 - 8q = 11$ even which is a contradiction so p is odd ***R1***

Note: This ***R1*** should be awarded for any valid reason to conclude that p must be odd.

THEN

$p = 2k + 1(, k \in \mathbb{Z})$ ***M1***

$(2k + 1)^2 = 8q + 11$

$4k^2 + 4k + 1 = 8q + 11$ ***(A1)***

$4k^2 + 4k = 8q + 10$

$2k^2 + 2k = 4q + 5$ or equivalent with one side odd and one side even ***A1***

a contradiction as LHS is even and RHS is odd ***R1***

Note: This ***R1*** is dependent on all previous marks.
Accept correct variations such as work based on $p = 2k - 1$.

therefore, if $p, q \in \mathbb{Z}$ then $p^2 - 8q - 11 \neq 0$ ***AG***

Total [6 marks]

Section B

10. (a) recognition that $45 = 10 + 10 + \text{arc length}$ ***(M1)***

arc length $= 25$ (cm) ***(A1)***

$25 = 12\theta$ ***A1***

$\theta = 2.08$ correct to 3 significant figures ***AG***

[3 marks]

(b)

> **Note:** There are many different ways to dissect the cross-section to determine its area. In all approaches, candidates will need to find w or $\frac{w}{2}$. Award the first three marks for work seen anywhere.

EITHER

evidence of using the cosine rule OR sine rule ***(M1)***

$w^2 = 12^2 + 12^2 - 2 \cdot 12 \cdot 12 \cos(2.08)$ OR $\frac{w}{\sin(2.08)} = \frac{12}{\sin(0.530796...)}$ ***(A1)***

$w = 20.6977\ldots$ or $\frac{w}{2} = 10.3488...$ ***(A1)***

OR

using trig ratios in a right triangle with angle $\frac{2.08}{2}$ and side length $\frac{w}{2}$ ***(M1)***

$\sin\left(\frac{2.08}{2}\right) = \frac{\frac{w}{2}}{12}$ ***(A1)***

$w = 20.6977\ldots$ or $\frac{w}{2} = 10.3488...$ ***(A1)***

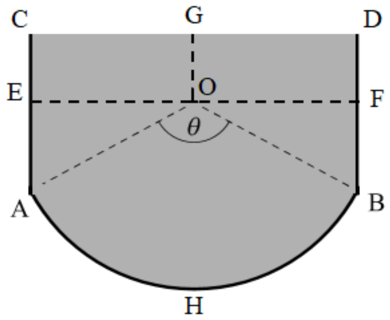
> **Note:** Accept $w = 20.7179\ldots$ from use of $\frac{\theta}{2} = \frac{25}{24}$.

continued…

Question 10 continued

THEN

Let the points A, B, C, D, E, F, G, H lie on the figure as follows:



EIGHER

(segment AHB =) sector OAB – triangle OAB ***(M1)***

$= \frac{1}{2}\times 12^2 \times 2.08 - \frac{1}{2}\times 12^2 \times \sin 2.08 \left(= 149.76 - 62.8655... = 86.8944...\right)$ ***(A1)***

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

sector OAB – triangle OAB + rectangle CDBA

$= 86.8944... + 10w \left(= 86.8944... + 206.977...\right)$

Note:	Use of $\theta = \frac{25}{12}$ throughout leads to segment OAB $= 87.2517...$ and cross-sectional area $= 87.2517... + 207.179...$.

continued…

Question 10 continued

OR

trapezium CGOA (= rectangle CGOE + triangle EOA) ***(M1)***

$=\frac{1}{2}\times\left(10+\left(10-12\cos(1.04)\right)\right)\times\frac{20.6977...}{2}\ (=72.0557)$ ***(A1)***

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

2 × trapezium CGOA + sector OAB

$= 2(72.0557...)+\frac{1}{2}\times12^2\times2.08\left(=144.111...+149.76\right)$

Note:	Use of $\theta=\frac{25}{12}$ leads to area of trapezium CGOA $=72.2154...$ and cross-sectional area $=144.430...+150$.

OR

2 x area of trapezium CGOA (= area of rectangle CDFE + 2 x triangle EOA) ***(M1)***

$20.6977...\times\left(10-12\cos(1.04)\right)+2\times\frac{1}{2}\times12\cos(1.04)\times12\sin(1.04)$ ***(A1)***

$\left(=81.2458...+62.8655...\right)$

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

2 x trapezium CGOA + sector OAB

$=144.111...+\frac{1}{2}\times12^2\times2.08\ \left(=144.111...+149.76\right)$

Note:	Use of $\theta=\frac{25}{12}$ leads to 2 x area of trapezium CGOA $=144.430...$ and cross-sectional area $=144.430...+150$.

THEN

area of cross-section $=293.871...$ ($294.430...$ from exact answer)

$=294$ (cm^2) ***A1***

[7 marks]

continued…

Question 10 continued

(c) **METHOD 1**

$-4.71976...$ volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognising rainfall can be represented by an integral ***(M1)***

$\int_0^{60} R'(t)\,\mathrm{d}t \quad \left(= \frac{250}{2\pi}\sin\left(\frac{2\pi\times 60}{5}\right) + 3000\times 60\right)$ ***(A1)***

Note: Accept any 60 second interval or any interval which is a multiple of 5 seconds (one period) scaled up to 60 seconds e.g. $12\int_0^{5} R'(t)\,\mathrm{d}t$.

rainfall over 60 seconds $=180000$ (cm^3) ***A1***

the gutter will overflow because the rainfall > gutter volume ***A1***

METHOD 2

volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognition that cosine has a minimum value of -1 ***(M1)***

$R'(t) \geq -1\times 50 + 3000\left(\text{cm}^3\text{s}^{-1}\right)$ ***(A1)***

rainfall over 60 seconds ≥ 177000 ***(A1)***

the gutter will overflow because the rainfall > gutter volume ***A1***

METHOD 3

volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognising rainfall can be represented by an integral ***(M1)***

attempt to solve $60 > 58.8$ OR $\int_0^{T} R'(t)\,\mathrm{d}t = 176658$ ***(M1)***

time to reach overflow point = $58.7875...$ OR $58.8990...$ ***A1***

the gutter will overflow because $60 > 58.8$ OR $60 > 58.9$ ***A1***

[5 marks]

Total [15 marks]

11. (a) $\mathrm{E}(X)=\int_0^2 \frac{6x}{\pi\sqrt{16-x^2}}\,\mathrm{d}x$ ***(A1)***

Note: Condone the absence of $\mathrm{d}x$.
Accept $\int_0^2 xf(x)\,\mathrm{d}x$

attempt to integrate $\frac{6x}{\pi\sqrt{16-x^2}}$ using inspection/substitution ***(M1)***

$-\frac{6}{2\pi}\int -2x\left(16-x^2\right)^{-\frac{1}{2}}\mathrm{d}x$ or let $u=16-x^2$

$-\frac{6}{2\pi}\left[2\left(16-x^2\right)^{\frac{1}{2}}\right]_0^2$ OR $\frac{6}{\pi}\left[-u^{\frac{1}{2}}\right]_{16}^{12}$ ***A1***

Note: For this ***A1*** condone absent or incorrect limits.

attempt to substitute their limits and evaluate ***(M1)***

$\frac{24}{\pi}-\frac{6}{\pi}\sqrt{12}\ \left(=\frac{12}{\pi}\left(2-\sqrt{3}\right)\right)$ ***A1***

Note: The substitution $\sin\theta=\frac{x}{4}$ may also be used, leading to
$\frac{24}{\pi}\int_0^{\frac{\pi}{6}}\sin\theta\,\mathrm{d}\theta=\frac{24}{\pi}\left[-\cos\theta\right]_0^{\frac{\pi}{6}}=\frac{24}{\pi}\left(1-\cos\frac{\pi}{6}\right)$. Award marks as appropriate and accept $\frac{24}{\pi}\left(1-\cos\frac{\pi}{6}\right)$ for the final ***A1***.

[5 marks]

(b) $\int_0^{0.5} f(x)\,\mathrm{d}x\left(=\int_0^{0.5}\frac{6}{\pi\sqrt{16-x^2}}\,\mathrm{d}x\right)$ ***(M1)***

$P(X<0.5)=0.239358...$

$=0.239$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) **EITHER**

recognition P(at least one success after n trials) = 1 – P(no successes after n trials) ***(M1)***

$1-\left(1-0.239...\right)^{n}\geq 0.99$ ***(A1)***

$n=16.8321...$

Note: Use of 0.239 results in $n=16.8612...$

OR

recognition that $Y\sim \mathrm{B}\left(n, 0.239...\right)$ ***(M1)***

If $n=16$ P(at least one success after n trials) $=0.987443...$

and if $n=17$ P(at least one success after n trials) $=0.990448...$ ***(A1)***

Note: Use of 0.239 results in the values $0.987348...$ and $0.990371...$

THEN

17 trials ***A1***

[3 marks]

(d) recognition that $Y\sim \mathrm{B}\left(10, \text{their part b)}\right)$ ***(M1)***

$\mathrm{B}\left(10, 0.239...\right)$

$\mathrm{P}\left(X=3\right)=0.242430...$

$=0.242$ ***A1***

[2 marks]

continued…

Question 11 continued

(e) 8 ***A1***

[1 mark]

(f) (i) $n-2$ ***A1***

(ii) ${}^{n}C_{3}$ (ways of 3 successes in n trials) ***(A1)***

$\frac{n-2}{{}^{n}C_{3}}$ ***(A1)***

Attempt to solve their $\frac{n-2}{{}^{n}C_{3}} > 0.05$ OR $\frac{6}{n(n-1)} > 0.05$ (or equivalent) ***(M1)***

Note: Accept an equation.

$n = 11.4658...$ OR

table values $n = 11, \frac{n-2}{{}^{n}C_{3}} = 0.0545454...$ and $n = 12, \frac{n-2}{{}^{n}C_{3}} = 0.0454545...$ ***(A1)***

$n = 11$ ***A1***

[6 marks]

Total [19 marks]

12.

Note: Penalise only once for an answer not given to six significant figures in parts (a), c(ii) and (d)(ii).

(a) attempt to use Euler's method ***(M1)***

$y_{n+1} = y_n + 0.03\left(\frac{x_n^{\ 2} y_n - y_n}{x_n^{\ 2} + 1}\right), y_1 = 3 + 0.03\left(\frac{0-3}{0+1}\right)$

$y_1 = 2.91$ ***(A1)***

at least one **correct** further intermediate value given to at least 3 significant figures ***(A1)***

y_0	3
y_1	2.91
y_2	2.82285...
y_3	2.73877...
y_4	2.65793...
y_5	2.58046...

$y(0.15) \approx y_5 = 2.58046160...$

$= 2.58046$ ***A1***

Note: Award final ***A1*** for the correct answer seen as the last line in a table. If the table goes beyond this value and the correct answer is not explicitly identified award maximum ***(M1)(A1)(A1)A0***

[4 marks]

continued…

Question 12 continued

(b) (i) $\frac{dy}{dx} = -3$ ***A1***

(ii) **METHOD 1**

attempt to use quotient (or product) rule on

$\frac{dy}{dx} = \frac{x^2 y - y}{x^2 + 1}\left(= \left(x^2+1\right)^{-1}\left(x^2 y - y\right)\right)$ ***(M1)***

attempt to use product rule and implicit differentiation on $x^2 y$ ***(M1)***

$\frac{d}{dx}\left(x^2 y - y\right) = x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}$ (seen anywhere) ***A1***

$= 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$)

$$\frac{d^2 y}{dx^2} = \frac{\left(x^2+1\right)\left(x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}\right) - \left(x^2 y - y\right)(2x)}{\left(x^2+1\right)^2}$$

OR $\frac{d^2 y}{dx^2} = \left(x^2+1\right)^{-1}\left(x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}\right) - 2x\left(x^2+1\right)^{-2}\left(x^2 y - y\right)$ ***A1***

$\frac{d^2 y}{dx^2} = 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$) ***AG***

METHOD 2

$\left(x^2+1\right)\frac{dy}{dx} = x^2 y - y$ ***A1***

attempt to use product rule and implicit differentiation ***(M1)***

$2x\frac{dy}{dx} + \left(x^2+1\right)\frac{d^2 y}{dx^2} = x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$\frac{d^2 y}{dx^2} = 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$) ***AG***

[5 marks]

continued…

Question 12 continued

(c) (i) $3-3x+\frac{x^2}{2!}(3)+\frac{x^3}{3!}(9)+\ldots\left(=3-3x+\frac{3}{2}x^2+\frac{3}{2}x^3+\ldots\right)$ ***A1A1***

Note: Award ***A1*** for first three terms, ***A1*** for fourth term

(ii) $y(0.15)=2.58881125\ldots$

$=2.58881$ ***A1***

[3 marks]

continued…

Question 12 continued

(d) (i) **EITHER**

attempt to separate variables ***M1***

$\int \frac{1}{y}\mathrm{d}y = \int \frac{x^2-1}{x^2+1}\mathrm{d}x$ OR $\int \frac{1}{y}\mathrm{d}y = \int \left(1-\frac{2}{x^2+1}\right)\mathrm{d}x$

$\ln y = x - 2\arctan x + c$ ***A1A1***

Note: Award ***A1*** for $\ln y$ or $\ln|y|$, ***A1*** for $x-2\arctan x$.
Condone missing $+c$ at this stage.

OR

attempt to use integrating factor ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} - y\left(1-\frac{2}{x^2+1}\right) = 0$

IF $= \mathrm{e}^{\int -\left(1-\frac{2}{x^2+1}\right)\mathrm{d}x} = \mathrm{e}^{-x+2\arctan x}$ ***A1***

$\mathrm{e}^{-x+2\arctan x}\frac{\mathrm{d}y}{\mathrm{d}x} - y\mathrm{e}^{-x+2\arctan x}\left(1-\frac{2}{x^2+1}\right) = 0$

$\mathrm{e}^{-x+2\arctan x}y = A$ ***A1***

THEN

$\ln y = x - 2\arctan x + c$ OR $y = A\mathrm{e}^{x-2\arctan x}$

attempt to find c or A using $x=0, y=3$ ***(M1)***

$\ln 3 = 0 - 2\arctan 0 + c$ OR $3 = A\mathrm{e}^{0-2\arctan 0}$

$c = \ln 3$ OR $A = 3$ ***(A1)***

Note: This ***A1*** should not be awarded if a correct value of c or A is preceded by incorrect working.

$y = \mathrm{e}^{x-2\arctan x+\ln 3}\left(=3\mathrm{e}^{x-2\arctan x}\right)$ ***A1***

continued…

Question 12 continued

(ii) $y(0.15) = 2.58786288...$

$= 2.58786$ ***A1***

[7 marks]

(e) the graph of $y = f(x)$ is concave up OR $\frac{d^2y}{dx^2} > 0$ (for $0 \le x < 1$) ***A1***

Note: Allow positive curvature, opening upwards, increasing first derivative.

hence tangents used (in Euler's method) give an underestimate,

so the approximate value for y when $x = 1.5$ is less than the actual value. ***R1***

[2 marks]

Note: ***R1*** is dependent on ***A1***, as well as reference to tangents, in words or on a diagram.

Total [21 marks]